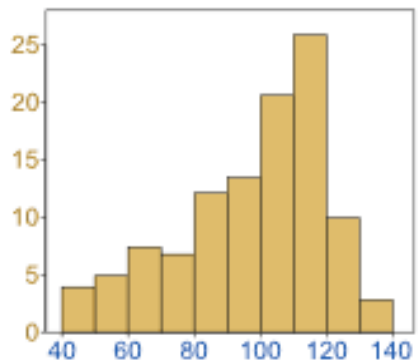
Introduction to the Normal Distribution

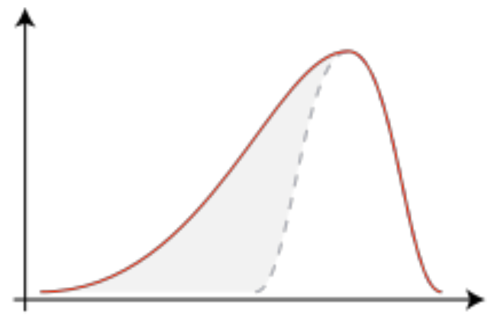
A. Recall… Histograms!

For a continuous variable, we can create *intervals*, populate a *frequency table*, and graph the resulting frequencies as a ***histogram*** or a ***polygon***.

A histogram can be “distributed” (spread out) in different ways. For example:

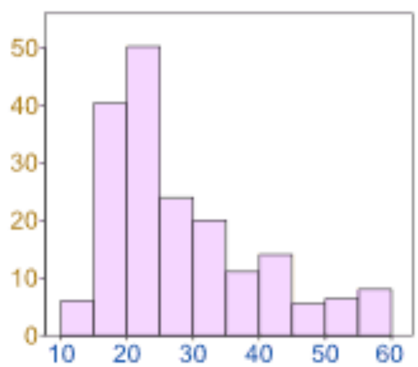
1. it can have a “tail” going to the left with one mode (called ***Negative Skew***) …

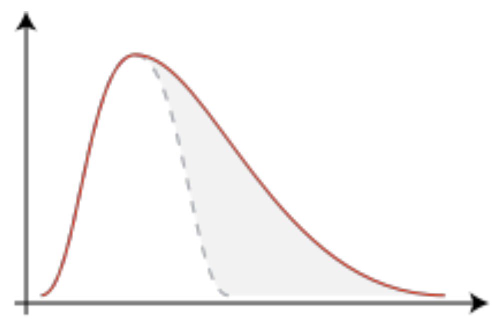




Negative Skew

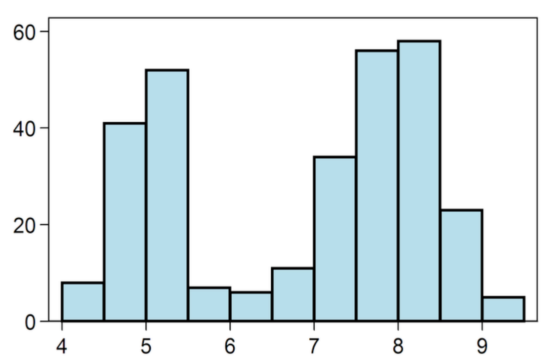
2. it can have a “tail” going to the right with one mode (called ***Positive Skew***) …

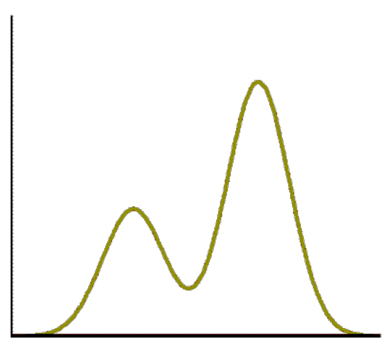




Positive Skew

3. it can have more than one mode (***Bimodal***, etc.) …

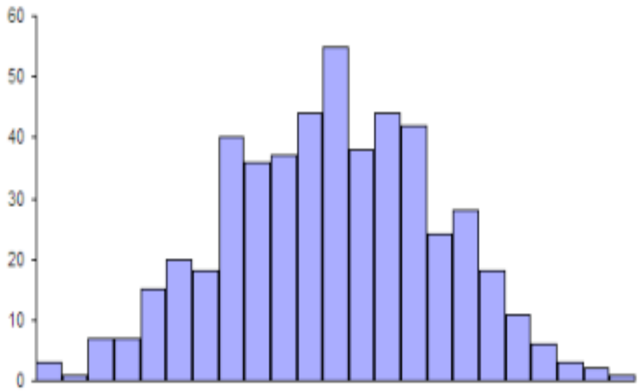


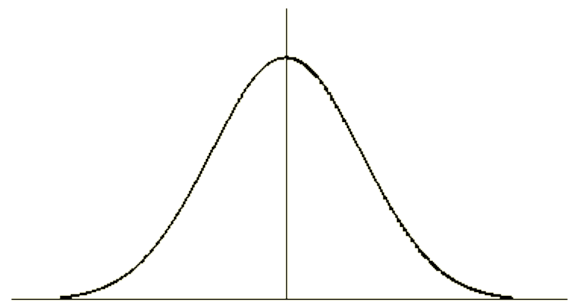


Bimodal

Images from: Pierce, Rod. (17 Dec 2012). "Normal Distribution". Math Is Fun. Retrieved 10 Mar 2013 from http://www.mathsisfun.com/data/standard-normal-distribution.html

4. it can be ***symmetric*** about the mean …



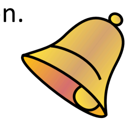


Symmetric

*There are many cases where the data is symmetric around a central value with no bias left or right. These cases happen so frequently, in fact, that this type of distribution is called a "**Normal Distribution**".*

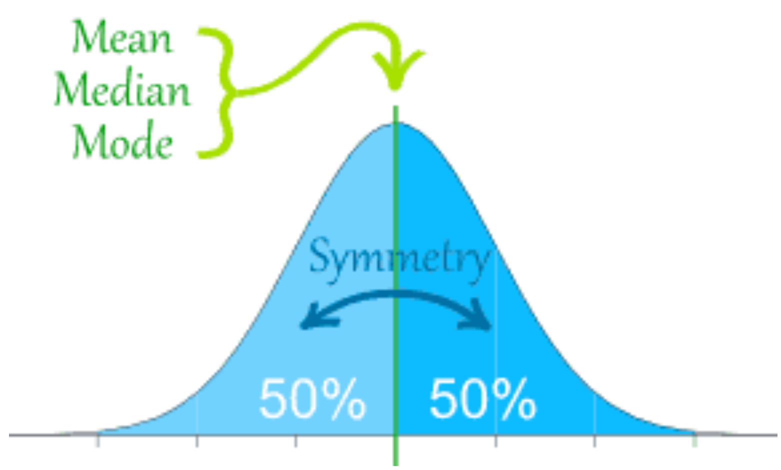
B. Properties of the Normal Distribution

Many physical quantities, like height and mass, are distributed symmetrically about the mean. Statisticians observe this "bell curve" so often that its mathematical model is known as the *normal* distribution. The physical, social and psychological sciences all make extensive use of normal models!

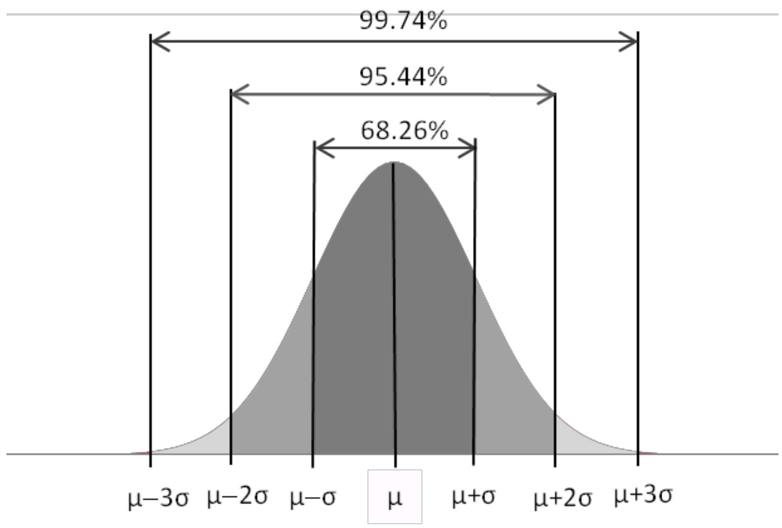


Cool facts about the normal distribution:

1. The symmetric, unimodal form of the distribution makes the **mode** equal the **median** equal the **mean**!
2. 50% of the data values lie below the mean and 50% of the data values lie above the mean.
3. The normal distribution is a *continuous* curve, so the area under the entire curve is equal to 1 or 100%.

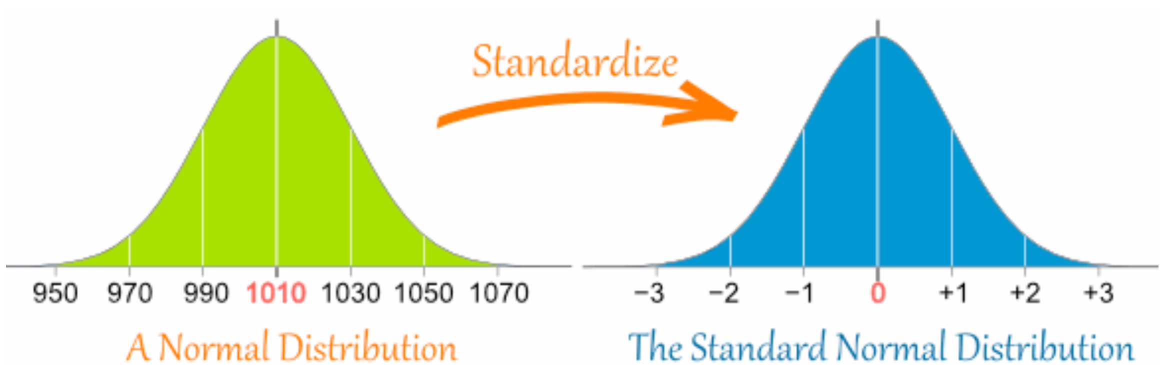


4. Although every normal distribution has its own *mean* (μ) and *standard deviation* (σ), for **any** normal distribution:
   - Approx 68% of the data will lie within 1 standard deviation of the mean (μ ± σ)
   - Approx 95% of the data will lie within 2 standard deviations of the mean (μ ± 2σ)
   - Approx 99.7% of the data will lie within 3 standard deviations of the mean (μ ± 3σ)



C. Standard Scores (z-scores)

The number of standard deviations any data value is from the mean is also called its "standard score" or "z-score". For data values below the mean, their z-scores will be negative; for data values above the mean, their z-scores will be positive. An entire normal curve can be "standardized" like the example below, which has a mean of 1010 and a standard deviation of 20:

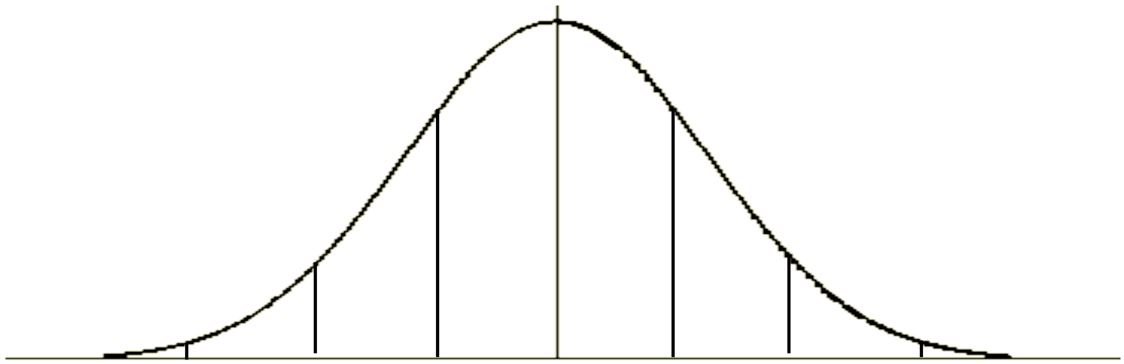


Computing z-scores:

For a Population:	• **z** is the "z-score" (Standard Score) • **x** is the value to be standardized • **μ** is the population mean • **σ** is the standard deviation
For a Sample:	• **z** is the "z-score" (Standard Score) • **x** is the value to be standardized • **xbar** is the sample mean • **s** is the standard deviation

Example: A survey of daily travel time was found to be *normally distributed*, with a mean travel time of 38.8 minutes and a standard deviation of 11.4 minutes.

a) Label a normal distribution curve for this data with the *survey scores* and the *standard scores*.



b) Standardize (find z-scores for) values of 26 minutes, 33 minutes, and 65 minutes to find out how many standard deviations each one is from the mean. Then, mark these points on the normal distribution you created in part a) above.

c) One commuter reported traveling 83 minutes to work. Values larger than 3 standard deviations from the mean are likely to be outliers. Does this traveler's time represent a possible outlier for this data?

CW: p. 430 #5; WORKSHEET: "Introduction to the Normal Distribution"

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1. Find the z-score corresponding to a grade of 52 from a normal distribution with a mean of 48 and a standard deviation of 1.8.

2. In North America, adult female heights have an approximate normal distribution with a mean of 65.0 inches and a standard deviation of 3.5 inches. Adult male heights have an approximate normal distribution with a mean of 70.0 inches and a standard deviation of 4.0 inches.

   What is your height in inches? ________________

   Find the z-score of your height. ________________

3. A certain brand of automobile tire has a life span of 35,000 miles and a standard deviation of 2250 miles. If the life spans of three randomly selected tires are: i) 34,000 miles, ii) 37,000 miles, and iii) 31,000 miles. Find the z-scores that correspond with each of these mileages. Would the life spans of any of the tires be considered unusual? Explain.

4. A highly selective university will only admit students who place at least 2-zscores above the mean on the ACT that has a mean of 18 and a standard deviation of 6. What is the minimum score that an applicant must obtain to be admitted to the university?

5. On a statistics test the class mean was 63 and the standard deviation was 7 and for the biology test the mean was 23 and standard deviation was 3.9.

Use z-scores to determine on which test each student did better.

a) Student A received a 73 on the stat test and a 26 on the biology test.

b) Student B received a 60 on the stat test and a 20 on the biology test.

c) Student C received a 70 on the stat test and a 29 on the biology test.

6. A manufacturer of bolts has a quality control policy that requires it to destroy any bolts that are more than 2 standard deviations from the mean. The quality control engineer knows that the bolts coming off the assembly line have a mean length of 8 cm with a standard deviation of 0.05 cm. For what length(s) will a bolt be destroyed?

7. A pharmaceutical company wants to test a new cholesterol drug. The average cholesterol of the target population is 200 mg and they have a standard deviation of 25 mg. The company wished to test a sample of people who fall between 1.5 and 3 z-scores above the mean. Into what range must a candidate's cholesterol level be in order for the candidate to be included in the study?

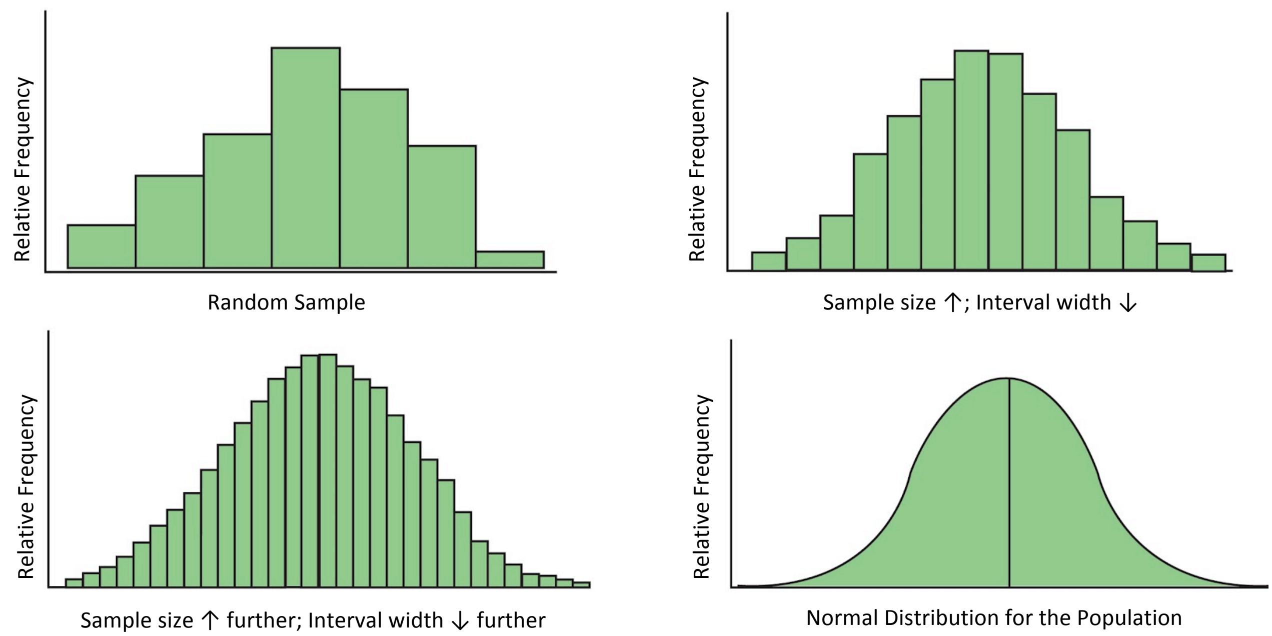
Answer Key: 1. $z = 2.22$ 2. Answers will vary. 3. i) $z = -0.44$, ii) $z = 0.89$, iii) $z = -1.78$ 4. Min. score is 30
5.a) stat: $z = 1.43$, bio: $z = 0.77$ ∴better on stat test 5.b) stat: $z = -0.43$, bio: $z = -0.77$ ∴better on stat test
5.c) stat: $z = 1.00$, bio: $z = 1.54$ ∴better on bio test 6. Bolts outside the interval [7.9, 8.1] will be destroyed
7. Candidates with cholesterol inside the interval [237.5, 275] will be included

Probabilities and the Normal Distribution

A. Probabilities and the Normal Distribution

A **probability distribution** involves the probabilities for *all possible outcomes* of an experiment. A **random variable** (usually uppercase *X*) must be defined, which has a single value for each outcome (usually lowercase *x*) in the experiment, i.e. P(X = *x*) = ______.

1. As the sample size approaches infinity, and the interval widths become smaller and smaller, the relative frequency polygon approximates a smooth curve.



2. The *relative frequency* (i.e. frequency ÷ total) for each interval is really the *probability* of finding a given value within that interval.
3. If we shrink the interval widths down to an infinitely small gap (i.e. they essentially represent discrete values for a continuous variable) then we have an ***infinite number of bars***, each with a width of essentially zero, making up the histogram!
4. This relative frequency polygon can now be referred to as a continuous probability distribution, where the area under the curve (i.e. the sum of all the bar heights) or the *cumulative relative frequency* equals 1 (100%).

B. Using the Normal Distribution and z-scores to Find Probabilities

- Because of the infinite number of possible outcomes, the probability of any ***individual*** outcome for a continuous random variable, X, is P(x) = _______________ = _____ .
- The probability of a ***range*** of outcomes, however, is defined. This means that we can calculate the probability that a given value will fall within a certain interval [e.g. P(X < 5)].
- The probability for a range of outcomes is simply the **area** between these values under the normal-distribution curve. These areas can be found on a **z-score table**.

I. Finding Probabilities *Less Than* a Given Value, P(X < a)

If we use the travel time survey results from last lesson, which had a mean of 38.8 minutes and a standard deviation of 11.4 minutes, we can find out the probability that a commuter will spend less than 52 minutes traveling to work.

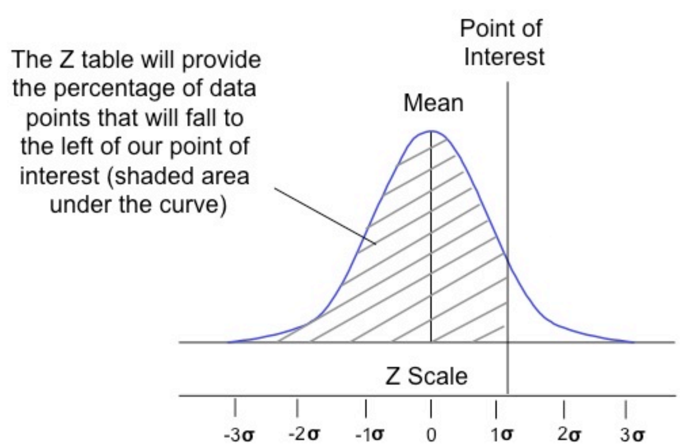
1. First we find the z-score corresponding to 52 minutes:

2. Then, we find this z-score on the table. ***Note*** that the outer row and column (shaded grey in your table) represent the z-score to two decimal places. Where the appropriate row and column for your z-score intersect is the probability of finding a value ***less than*** (i.e. to the left of) this in your data.

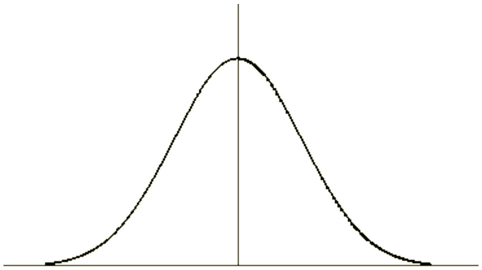
	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962

z-score second decimal place value (hundredths)

z-score up to the first decimal place value (ones and tenths)



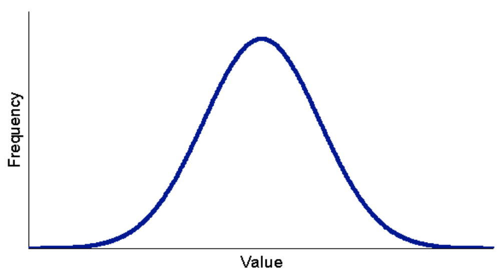
Ex. 1: The Research In Motion plant in Waterloo is producing the company's sim card cages. The widths of the cages are normally distributed with a mean of 95mm and a standard deviation of 2mm. What is the probability that the next sim card cage will have a width smaller than 93mm?



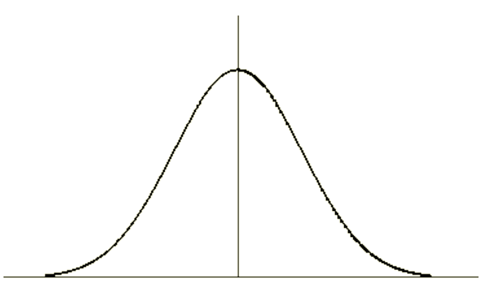
II. Finding Probabilities *Greater Than* a Given Value, P(X > a)

To find the probability of an outcome being **greater than** a given value, we find the z-score for the area below the curve for that value, and we subtract its corresponding probability from 1:

$P(X > a)$ = the probability that a random continuous variable X is greater than a
= the area under the curve above x = a
= $1 - P(X < a)$



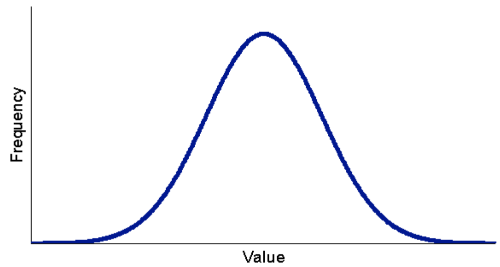
Ex. 2: Using z-scores, find the percentage of sim card cage widths that are *above* 98.2 mm.



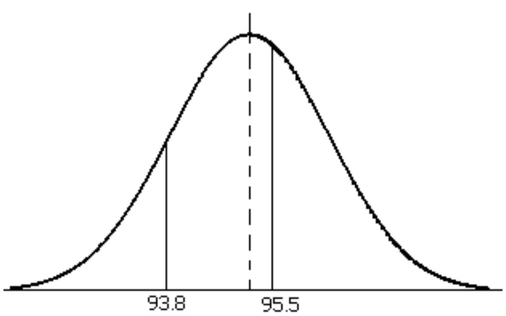
III. Finding Probabilities *Between* Given Values, P(a < X < b)

To find the probability of being **between two values** we find z-scores for both and subtract their probabilities:

P(a < X < b) = the probability that a random continuous variable X lies between the values a and b
= the area under the curve from x = a to x = b
= P(X < b) – P(X < a):



Ex. 3: Using z-scores, find the percentage of sim card cage widths that are *between* 93.8mm and 95.5mm.



CW: p. 430 #1, 2, 3, 4(see p. 428 *Example 3: Solution 2*), 6, 7, 8, 13; p. 439 #1, 3, 12($\bar{x}$ = 351.68, s = 3.33, P = 0.364)

Confidence Intervals

A. What's a Confidence Interval?

Newspapers, magazines and other forms of media will often give statistics on an issue and provide the level of certainty or accuracy associated with the information.

Recall: Suppose we want to report on the mean of our population. We collect a ***sample*** so that we can *estimate* the mean of our ***population***. If we conduct a survey that *is not* a census, we cannot truly determine the population mean, μ; instead, we can determine a **confidence interval** that highlights ranges of values within which μ is likely to fall. Because the normal distribution is symmetrical, these intervals are centered around the sample mean, $\bar{x}$. (*Note*: The closer *n* gets to *N*, and the smaller the standard deviation, the more confident we can be about our mean!)

Ex. 1: The statement, "We can conclude with 95% confidence that millionaires donate between 12.6% and 17.4% of their income to charity." It is saying that:

There is a _______% probability that millionaires donate between ________ and ________ of their income to charity.

Ex. 2: Interpret the statement: "A study showed that 60% of first-year post secondary enrollees do not come directly from high school. These results are accurate $\pm$5%, 19 times out of 20".

B. How do you create confidence intervals?

In this course, we will only be considering confidence intervals of data that follows a normal distribution. We will use the follow definition:

CONFIDENCE INTERVAL FOR A NORMAL DISTRIBUTION

$$\left(\bar{x} - z_{\frac{\alpha}{2}}\left(\frac{\sigma}{\sqrt{n}}\right),\ \bar{x} + z_{\frac{\alpha}{2}}\left(\frac{\sigma}{\sqrt{n}}\right)\right) \quad \text{or} \quad \bar{x} \pm z_{\frac{\alpha}{2}}\left(\frac{\sigma}{\sqrt{n}}\right)$$

The table gives a list of common confidence levels and their associated z-scores:

CONFIDENCE LEVEL	Z-SCORE, $z_{\frac{\alpha}{2}}$
90 %	1.645
95 %	1.960
99 %	2.576

Ex. 3: Suppose we conduct a survey of 19 millionaires to find out what percent of their income the average millionaire donates to charity. We discover that the mean percent is 15 with a standard deviation of 5 percent. Assuming the distribution of all charity percents is approximately normal:

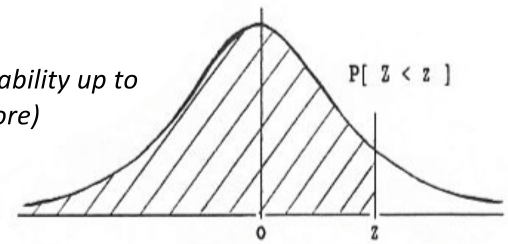
a) Find a 95% confidence interval for the mean percent.

b) Find a 99% confidence interval for the mean percent.

CW: p. 464 #1 – 4, 7, 8a

Z-score Table for Areas Under the Normal Distribution

This table gives the cumulative probability up to the standardized normal value (z-score)



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3829
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0..6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986