Uniform Probability Distributions

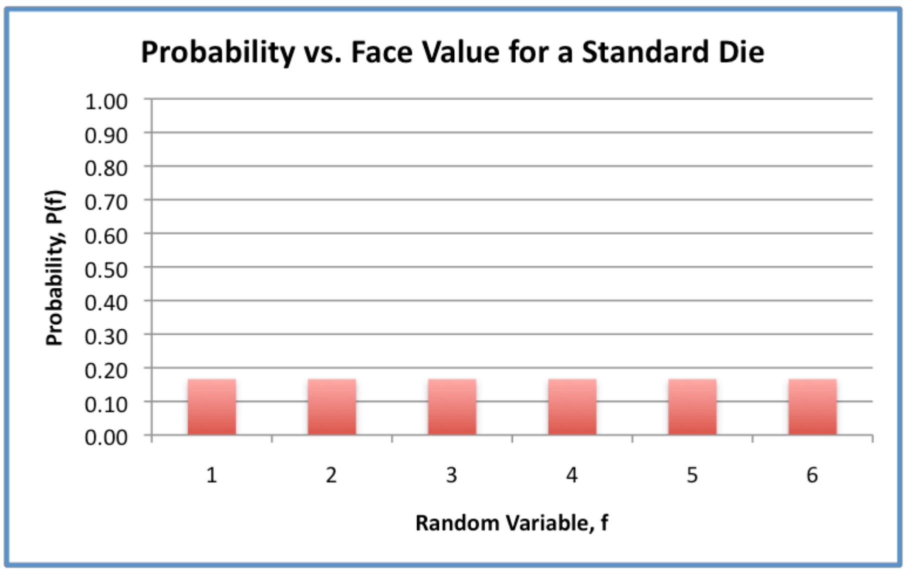
A. Probability Distributions

In the previous unit, we looked at probabilities of *individual outcomes* for an experiment. A **probability distribution** involves the probabilities for *all possible outcomes* of an experiment, often shown as a table or as a graph of *probability vs. the value of a random variable.*

For a probability experiment, a **random variable** must be defined, which has a single value for each of the outcomes in the experiment. For example, we may be looking at the random variable F, which is the "face-up number" displayed on a standard die. Each possible outcome for the random variable F is denoted as a lowercase f. In essence, uppercase F defines the random variable to be measured, while the lowercase f denotes the individual values (*i.e.* for the random variable F, we may be asked to find the probability when f = 1).

Ex. 1: Probability Distribution for *F*

Random Variable, *f*	Probability, *P(f)*



****Recall: The sum of all probabilities in a given distribution = _____**

If all outcomes in a distribution are *equally likely*, then it is called a **uniform probability distribution**.

Probability in a Discrete Uniform Distribution

In a discrete uniform distribution, all outcomes are equally likely, so for all values of x,

$$P(x) = \frac{1}{n}$$

where *n* is the number of possible outcomes in the experiment.

B. Random Variables

Most times, the random variables are defined as *X or Y*. Random variables can be **discrete** or **continuous**. **Discrete variables** can only take on *specific values* (usually integers) within a given range, while **continuous variables** have an *infinite number of possible values* in a continuous interval.

Ex. 2: Classify each of the following random variables as discrete or continuous:

a) the number of phone calls made by a sales person

b) the length of time the salesperson spent on the phone

c) the number of widgets sold by the company

d) the distance from the Earth to the sun

C. Expected Values

An **expectation** or **expected value,** E(X), for a distribution is the ***predicted*** **average** of all possible outcomes of a probability experiment. The expectation is equal to the sum of the products of each outcome (random variable = x_i) with its probability, $P(x_i)$

The **Expected Value, E(X),** is the predicted average of all possible outcomes of a probability experiment.

$$E(X) = x_1 \cdot P(x_1) + x_2 \cdot P(x_2) + \cdots + x_n \cdot P(x_n)$$
$$= \sum_{i=1}^{n} x_i \cdot P(x_i)$$

Ex. 3: Consider a simple game in which you roll a single die. If you roll an even number, you get that number of points, and if you roll an odd number, you lose that number of points. a) Set up a probability distribution of points in this game; b) calculate the expected value for the number of points per roll; c) Is this a fair game?

Number on Upper Face	**Points, x**	**Probability, *P(x)***

	Product of $x_i * P(x_i)$
E(X):	

Read p. 373 Example 4; HW: p. 374-377 # 1, 2, 3ac, 4, 5, 6ab, 8, 9, 12, 15, 18ab, 16 (use Excel for a and c)

Example 4 Canoe Lengths

A summer camp has seven 4.6-m canoes, ten 5.0-m canoes, four 5.2-m canoes, and four 6.1-m canoes. Canoes are assigned randomly for campers going on a canoe trip.

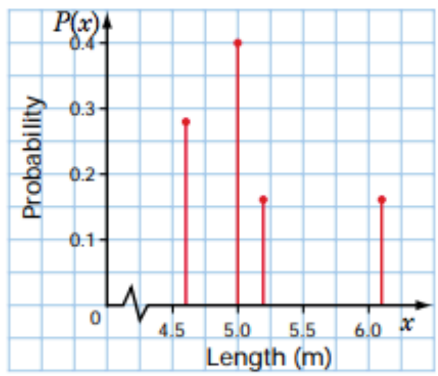
a) Show the probability distribution for the length of an assigned canoe.

b) What is the expected length of an assigned canoe?

Solution

a) Here the random variable is the canoe length.

Length of Canoe (m), x	Probability, $P(x)$
4.6	$\frac{7}{25}$
5.0	$\frac{10}{25}$
5.2	$\frac{4}{25}$
6.1	$\frac{4}{25}$



Observe that the sum of the probabilities is again 1, but the probabilities are not equal. This distribution is not uniform.

b)
$$E(X) = (4.6)\left(\frac{7}{25}\right) + (5.0)\left(\frac{10}{25}\right) + (5.2)\left(\frac{4}{25}\right) + (6.1)\left(\frac{4}{25}\right)$$
$$= 5.1$$

The expected length of the canoe is 5.1 m.

Key Concepts

- A random variable, X, has a single value for each outcome in the experiment. Discrete random variables have separated values while continuous random variables have an infinite number of outcomes along a continuous interval.
- A probability distribution shows the probabilities of all the possible outcomes of an experiment. The sum of the probabilities in any distribution is 1.
- Expectation, or the predicted average of all possible outcomes of a probability experiment, is

$$E(X) = x_1P(x_1) + x_2P(x_2) + \dots + x_nP(x_n) = \sum_{i=1}^{n} x_iP(x_i)$$

- The expected outcome in a fair game is 0.
- The outcomes of a uniform probability distribution all have the same probability, $P(x) = \frac{1}{n}$, where n is the number of possible outcomes in the experiment.
- You can simulate a probability distribution with manual methods, calculators, or computer software.

Communicate Your Understanding

1. Explain the principal differences between the graphs of the probability distributions in Example 2 and Example 4.
2. In the game of battleship, you select squares on a grid and your opponent tells you if you scored a "hit." Is this process a uniform distribution? What evidence can you provide to support your position?

Practise

A

1. Classify each of the following random variables as discrete or continuous.
   a) number of times you catch a ball in a baseball game
   b) length of time you play in a baseball game
   c) length of a car in centimetres
   d) number of red cars on the highway
   e) volume of water in a tank
   f) number of candies in a box
2. Explain whether each of the following experiments has a uniform probability distribution.
   a) selecting the winning number for a lottery
   b) selecting three people to attend a conference
   c) flipping a coin

d) generating a random number between 1 and 20 with a calculator

e) guessing a person's age

f) cutting a card from a well-shuffled deck

g) rolling a number with two dice

3. Given the following probability distributions, determine the expected values.

a)

x	$P(x)$
5	0.3
10	0.25
15	0.45

b)

x	$P(x)$
1 000	0.25
100 000	0.25
1 000 000	0.25
10 000 000	0.25

c)

x	$P(x)$
1	$\frac{1}{6}$
2	$\frac{1}{5}$
3	$\frac{1}{4}$
4	$\frac{1}{3}$
5	$\frac{1}{20}$

4. A spinner has eight equally-sized sectors, numbered 1 through 8.

a) What is the probability that the arrow on the spinner will stop on a prime number?

b) What is the expected outcome, to the nearest tenth?

Apply, Solve, Communicate

B

5. A survey company is randomly calling telephone numbers in your exchange.

a) Do these calls have a uniform distribution? Explain.

b) What is the probability that a particular telephone number will receive the next call?

c) What is the probability that the last four digits of the next number called will all be the same?

6. a) Determine the probability distribution for the sum rolled with two dice.

b) Determine the expected sum of two dice.

c) Repeat parts a) and b) for the sum of three dice.

7. There are only five perfectly symmetrical polyhedrons: the tetrahedron (4 faces), the cube (6 faces), the octahedron (8 faces), the dodecahedron (12 faces), and the icosahedron (20 faces). Calculate the expected value for dies made in each of these shapes.

8. A lottery has a \$1 000 000 first prize, a \$25 000 second prize, and five \$1000 third prizes. A total of 2 000 000 tickets are sold.

a) What is the probability of winning a prize in this lottery?

b) If a ticket costs \$2.00, what is the expected profit per ticket?

9. Communication A game consists of rolling a die. If an even number shows, you receive double the value of the upper face in points. If an odd number shows, you lose points equivalent to triple the value of the upper face.

a) What is the expectation?

b) Is this game fair? Explain.

10. Application In a lottery, there are 2 000 000 tickets to be sold. The prizes are as follows:

Prize (\$)	Number of Prizes
1 000 000	1
50 000	5
1 000	10
50	50

What should the lottery operators charge per ticket in order to make a 40% profit?

11. In a family with two children, determine the probability distribution for the number of girls. What is the expected number of girls?

12. A computer has been programmed to draw a rectangle with perimeter of 24 cm. The program randomly chooses integer lengths. What is the expected area of the rectangle?

13. Suppose you are designing a board game with a rule that players who land on a particular square must roll two dice to determine where they move next. Players move ahead five squares for a roll with a sum of 7 and three squares for a sum of 4 or 10. Players move back n squares for any other roll.

a) Develop a simulation to determine the value of n for which the expected move is zero squares.

b) Use the probability distribution to verify that the value of n from your simulation does produce an expected move of zero squares.

14. Inquiry/Problem Solving Cheryl and Fatima each have two children. Cheryl's oldest child is a boy, and Fatima has at least one son.

a) Develop a simulation to determine whether Cheryl or Fatima has the greater probability of having two sons.

b) Use the techniques of this section to verify the results of your simulation.

15. Suppose you buy four boxes of the Krakked Korn cereal. Remember that each box has an equal probability of containing any one of the seven collector cards.

a) What is the probability of getting

i) four identical cards?

ii) three identical cards?

iii) two identical and two different cards?

iv) two pairs of identical cards?

v) four different cards?

b) Sketch a probability distribution for the number of different cards you might find in the four boxes of cereal.

c) Is the distribution in part b) uniform?

ACHIEVEMENT CHECK

Knowledge/Understanding | Thinking/Inquiry/Problem Solving | Communication | Application

16. A spinner with five regions is used in a game. The probabilities of the regions are

$P(1) = 0.3$
$P(2) = 0.2$
$P(3) = 0.1$
$P(4) = 0.1$
$P(5) = 0.3$

a) Sketch and label a spinner that will generate these probabilities.

b) The rules of the game are as follows: If you spin and land on an even number, you receive double that number of points. If you land on an odd number, you lose that number of points. What is the expected number of points a player will win or lose?

c) Sketch a graph of the probability distribution for this game.

d) Show that this game is not fair. Explain in words.

e) Alter the game to make it fair. Prove mathematically that your version is fair.

17. **Application** The door prizes at a dance are gift certificates from local merchants. There are four \$10 certificates, five \$20 certificates, and three \$50 certificates. The prize envelopes are mixed together in a bag and are drawn at random.

a) Use a tree diagram to illustrate the possible outcomes for selecting the first two prizes to be given out.

b) Determine the probability distribution for the number of \$20 certificates in the first two prizes drawn.

c) What is the probability that exactly three of the first five prizes selected will be \$10 certificates?

d) What is the expected number of \$10 certificates among the first five prizes drawn?

C

18. Most casinos have roulette wheels. In North America, these wheels have 38 slots, numbered 1 to 36, 0, and 00. The 0 and 00 slots are coloured green. Half of the remaining slots are red and the other half are black. A ball rolls around the wheel and players bet on which slot the ball will stop in. If a player guesses correctly, the casino pays out according to the type of bet.

a) Calculate the house advantage, which is the casino's profit, as a percent of the total amount wagered for each of the following bets. Assume that players place their bets randomly.

i) single number bet, payout ratio 35:1

ii) red number bet, payout ratio 1:1

iii) odd number bet, payout ratio 1:1

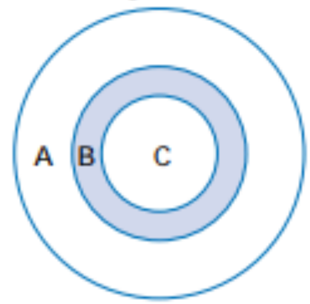
iv) 6-number group, payout ratio 5:1

v) 12-number group, payout ratio 2:1

b) Estimate the weekly profit that a roulette wheel could make for a casino. List the assumptions you have to make for your calculation.

c) European roulette wheels have only one zero. Describe how this difference would affect the house advantage.

19. **Inquiry/Problem Solving** Three concentric circles are drawn with radii of 8 cm, 12 cm, and 20 cm. If a dart lands randomly on this target, what are the probabilities of it landing in each region?



20. A die is a random device for which each possible value of the random variable has a probability of $\frac{1}{6}$. Design a random device with the probabilities listed below and determine the expectation for each device. Use a different type of device in parts a) and b).

a) $P(0) = \frac{1}{4}$

$P(1) = \frac{1}{6}$

$P(2) = P(3) = \frac{1}{8}$

$P(4) = P(5) = P(6) = P(7) = \frac{1}{12}$

b) $P(0) = \frac{1}{6}$

$P(1) = P(2) = \frac{1}{4}$

$P(3) = \frac{1}{3}$

21. **Communication** Explain how the population mean, μ, and the expectation, $E(X)$, are related.

Binomial Probability Distributions

A. What is a Binomial Distribution?

A **binomial experiment** is one that possesses the following properties:

1. The experiment consists of *n* repeated trials called *Bernoulli trials*;
2. Each trial results in an outcome that may be classified as a **success** or a **failure** (hence the name, **binomial**);
3. The probability of a success, denoted by *p*, remains *constant* from trial to trial, and repeated trials are *independent*.

The number of successes, X , in *n* trials of a binomial experiment is called a **binomial random variable**.

The probability of *x* successes in *n* binomial (or Bernoulli) trials is given by the formula:

$$P(x) = ({}_nC_x)p^x q^{(n - x)}$$

Where:

n = the number of trials
x = 0, 1, 2, ... n
p = the probability of success in a single trial
q = the probability of failure in a single trial (*i.e.* q = 1 – p)
${}_nC_x$ is a combination

The **binomial probability distribution** can be created by finding the probabilities for all values of *x* from 0 to *n*.

Ex. 1: A die is tossed 3 times. What is the probability of:

(a) no fives turning up?

(b) 1 five?

(c) 3 fives?

(d) Create a probability distribution for this experiment and find the expected value, *E(X).*

Number of Fives, *x*	**Probability, *P(x)***

	Product of $x_i * P(x_i)$
E(X):	

B. Expected Values

Since the probability of success for each trial is the same, the **expectation** or **expected value,** *E(X),* for a binomial distribution is the (*probability of success for each trial)* × (*the number of trials)*

The **expected value** for *n* binomial (or Bernoulli) trials is given by the formula:

$$E(x) = np$$

Where *p* is the probability of success for each independent trial

Ex. 2: Hospital records show that of patients suffering from a certain disease, 75% die as a result of it.

(a) What is the probability that of 6 randomly selected patients, 4 will recover?

(b) What is the expected number to recover in this sample?

Ex. 3: In the old days, there was a probability of 0.8 of success in any attempt to make a telephone call. Calculate the probability of having 7 successes in 10 attempts.

Ex. 4: A (blindfolded) marksman finds that on the average he hits the target 4 times out of 5. If he fires 4 shots, what is the probability of:

(a) more than 2 hits?

(b) at least 3 misses?

Ex. 5: A manufacturer of metal pistons finds that on the average, 12% of his pistons are rejected because they are either oversize or undersize. What is the probability that a batch of 10 pistons will contain

(a) no more than 2 rejects?

(b) at least 2 rejects?

HW: p. 385-387 # 2, 3, 5, 6bc, 7, 8ab, 9, 13acde, 17

Communicate Your Understanding

1. Consider this question: If five cards are dealt from a standard deck, what is the probability that two of the cards are the ace and king of spades?
   a) Explain why the binomial distribution is not a suitable model for this scenario.
   b) How could you change the scenario so that it does fit a binomial distribution? What attributes of a binomial distribution would you use in your modelling?
2. Describe how the graph in Example 2 differs from the graph of a uniform distribution.
3. Compare your results from the simulation of the Choco-Latie candies at the beginning of this section with the calculated values in Example 3. Explain any similarities or differences.

Practise

A

1. Which of the following situations can be modelled by a binomial distribution? Justify your answers.
   a) A child rolls a die ten times and counts the number of 3s.
   b) The first player in a free-throw basketball competition has a free-throw success rate of 88.4%. A second player takes over when the first player misses the basket.
   c) A farmer gives 12 of the 200 cattle in a herd an antibiotic. The farmer then selects 10 cattle at random to test for infections to see if the antibiotic was effective.
   d) A factory producing electric motors has a 0.2% defect rate. A quality-control inspector needs to determine the expected number of motors that would fail in a day's production.
2. Prepare a table and a graph for a binomial distribution with
   a) $p = 0.2$, $n = 5$
   b) $p = 0.5$, $n = 8$

Apply, Solve, Communicate

B

3. Suppose that 5% of the first batch of engines off a new production line have flaws. An inspector randomly selects six engines for testing.
   a) Show the probability distribution for the number of flawed engines in the sample.
   b) What is the expected number of flawed engines in the sample?
4. **Application** Design a simulation to predict the expected number of 7s in Tan's new telephone number in Example 2.
5. The faces of a 12-sided die are numbered from 1 to 12. What is the probability of rolling 9 at least twice in ten tries?
6. **Application** A certain type of rocket has a failure rate of 1.5%.
   a) Design a simulation to illustrate the expected number of failures in 100 launches.
   b) Use the methods developed in this section to determine the probability of fewer than 4 failures in 100 launches.
   c) What is the expected number of failures in 100 launches of the rocket?

7. Suppose that 65% of the families in a town own computers. If eight families are surveyed at random,

a) what is the probability that at least four own computers?

b) what is the expected number of families with computers?

8. Inquiry/Problem Solving Ten percent of a country's population are left-handed.

a) What is the probability that 5 people in a group of 20 are left handed?

b) What is the expected number of left-handed people in a group of 20?

c) Design a simulation to show that the expectation calculated in part b) is accurate.

9. Inquiry/Problem Solving Suppose that Bayanisthol, a new drug, is effective in 65% of clinical trials. Design a problem involving this drug that would fit a binomial distribution. Then, provide a solution to your problem.

10. Pythag-Air-US Airlines has determined that 5% of its customers do not show up for their flights. If a passenger is bumped off a flight because of overbooking, the airline pays the customer $200. What is the expected payout by the airline, if it overbooks a 240-seat airplane by 5%?

11. A department-store promotion involves scratching four boxes on a card to reveal randomly printed letters from A to F. The discount is 10% for each A revealed, 5% for each B revealed, and 1% for the other four letters. What is the expected discount for this promotion?

12. a) Expand the following binomials.

i) $(p + q)^6$ ii) $(0.2 + 0.8)^5$

b) Use the expansions to show how the binomial theorem is related to the binomial probability distribution.

ACHIEVEMENT CHECK

Knowledge/ Understanding	Thinking/Inquiry/ Problem Solving	Communication	Application

13. Your local newspaper publishes an Ultimate Trivia Contest with 12 extremely difficult questions, each having 4 possible answers. You have no idea what the correct answers are, so you make a guess for each question.

a) Explain why this situation can be modelled by a binomial distribution.

b) Use a simulation to predict the expected number of correct answers.

c) Verify your prediction mathematically.

d) What is the probability that you will get at least 6 answers correct?

e) What is the probability that you will get fewer than 2 answers correct?

f) Describe how the graph of this distribution would change if the number of possible answers for each question increases or decreases.

C

14. The French mathematician Simeon-Denis Poisson (1781–1840) developed what is now known as the *Poisson distribution*. This distribution can be used to approximate the binomial distribution if p is very small and n is very large. It uses the formula

$$P(x) = \frac{e^{-np}(np)^x}{x!},$$

where e is the irrational number 2.718 28 … (the base for the natural logarithm).

Use the Poisson distribution to approximate the following situations. Compare the results to those found using the binomial distribution.

a) A certain drug is effective in 98% of cases. If 2000 patients are selected at random, what is the probability that the drug was ineffective in exactly 10 cases?

b) Insurance tables indicate that there is a probability of 0.01 that a driver of a specific model of car will have an accident requiring hospitalization within a one-year period. If the insurance company has 4500 policies, what is the probability of fewer than 5 claims for accidents requiring hospitalization?

c) On election day, only 3% of the population voted for the Environment Party. If 1000 voters were selected at random, what is the probability that fewer than 8 of them voted for the Environment Party?

15. Communication Suppose heads occurs 15 times in 20 tosses of a coin. Do you think the coin is fair? Explain your reasoning.

16. Inquiry/Problem Solving

a) Develop a formula for $P(x)$ in a "trinomial" distribution that has three possible outcomes with probabilities p, q, and r, respectively.

b) Use your formula to determine the probability of rolling a 3 twice and a 5 four times in ten trials with a standard die.

17. Communication A judge in a model-airplane contest says that the probability of a model landing without damage is 0.798, so there is only "one chance in five" that any of the seven models in the finals will be damaged. Discuss the accuracy of the judge's statement.

Career Connection

Actuary

Actuaries are statistics specialists who use business, analytical, and mathematical skills to apply mathematical models to insurance, pensions, and other areas of finance. Actuaries assemble and analyse data and develop probability models for the risks and costs of accidents, sickness, death, pensions, unemployment, and so on. Governments and private companies use such models to determine pension contributions and fair prices for insurance premiums. Actuaries may also be called upon to provide legal evidence on the value of future earnings of an accident victim.

Actuaries must keep up-to-date on social issues, economic trends, business issues, and the law. Most actuaries have a degree in actuarial science, statistics, or mathematics and have studied statistics, calculus, algebra, operations research, numerical analysis, and interest theory. A strong background in business or economics is also useful.

Actuaries work for insurance companies, pension-management firms, accounting firms, labour unions, consulting groups, and federal and provincial governments.

WEB CONNECTION

www.mcgrawhill.ca/links/MDM12

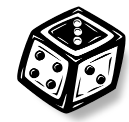
For more information about a career as an actuary, visit the above web site and follow the links.

Geometric Distributions

A. What is a Geometric Distribution?

Like binomial distributions, **geometric distributions** have only two possible outcomes: **success or failure**, whose probabilities do not change from one trial to the next (∴ independent trials). However, the random variable for a geometric distribution is the ***waiting time*** – the number of *unsuccessful independent trials before success occurs*.

Examples: a) In a board game, you cannot make your first move until a specific number is rolled on the die. This could take any number of trials before success is achieved.
b) Manufacturers of products need to know how many operations their products can perform before failing.



Order matters for geometric distributions, so we do not calculate the **combinations** for our single successful outcome!

Probability in a Geometric Distribution

The probability of success after a *waiting time of x* failures is given by the formula:

$$P(x) = q^{x}p$$

Where p is the probability of success in each single trial and q is the probability of failure

The **geometric probability distribution** can be created by finding the probabilities for all values of x from 0 to ∞! (This is not practical for *us* to do, and the probabilities begin converge on __________the greater the value of x.)

Ex. 1: To make your first move in a board game, you must roll a 6 on a standard die. What is the probability that you get a 6:

(a) on your first roll?

(b) on your 4th roll?

(c) within 3 rolls?

B. Expected Values

The **expected value,** *E(X),* for a geometric distribution is the *amount of wait time* (# of failures) before success, and it is the sum of an infinite series! However, it is possible to show that the expected value converges to a simple formula:

The **expected value (amount of wait time)** for a geometric distribution is given by the formula:

$$E(x) = \frac{q}{p}$$

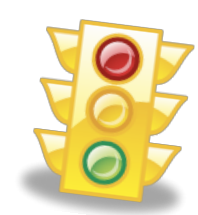
Where p is the probability of success and q is the probability of failure for each independent trial

Ex. 2: A basketball player has a success rate of 68% for scoring on free throws. What is the expected waiting time before the player misses the basket on a free throw?

Ex. 3: Suppose you pass a traffic light on your way to school each day that alternates being green for 40 s and then amber or red for 60 s.

a) What is the probability that the light will be green at least once per week when you reach the intersection?

b) What is the expected number of days you must reach the intersection before you get a green light?



Complete ***Example #1*** **on p. 389-391 using Excel; HW: p. 394-396 # 1, 2, 3, 5, 7, 8-13**

Practise

A

1. Which of these random variables have a hypergeometric distribution? Explain why.
   a) the number of clubs dealt from a deck
   b) the number of attempts before rolling a six with a die
   c) the number of 3s produced by a random-number generator
   d) the number of defective screws in a random sample of 20 taken from a production line that has a 2% defect rate
   e) the number of male names on a page selected at random from a telephone book
   f) the number of left-handed people in a group selected from the general population
   g) the number of left-handed people selected from a group comprised equally of left-handed and right-handed people

2. Prepare a table and a graph of a hypergeometric distribution with
   a) $n = 6, r = 3, a = 3$ b) $n = 8, r = 3, a = 5$

Apply, Solve, Communicate

B

3. There are five cats and seven dogs in a pet shop. Four pets are chosen at random for a visit to a children's hospital.
   a) What is the probability that exactly two of the pets will be dogs?
   b) What is the expected number of dogs chosen?

4. Communication Earlier this year, 520 seals were caught and tagged. On a recent survey, 30 out of 125 seals had been tagged.
   a) Estimate the size of the seal population.
   b) Explain why you cannot calculate the exact size of the seal population.

5. Of the 60 grade-12 students at a school, 45 are taking English. Suppose that 8 grade-12 students are selected at random for a survey.
   a) Develop a simulation to determine the probability that 5 of the selected students are studying English.
   b) Use the formulas developed in this section to verify your simulation results.

6. Inquiry/Problem Solving In a study of Canada geese, 200 of a known population of 1200 geese were caught and tagged. Later, another 50 geese were caught.
   a) Develop a simulation to determine the expected number of tagged geese in the second sample.
   b) Use the formulas developed in this section to verify your simulation results.

7. Application In a mathematics class of 20 students, 5 are bilingual. If the class is randomly divided into 4 project teams,
   a) what is the probability that a team has fewer than 2 bilingual students?
   b) what is the expected number of bilingual students on a team?

8. In a swim meet, there are 16 competitors, 5 of whom are from the Eastern Swim Club.
   a) What is the probability that 2 of the 5 swimmers in the first heat are from the Eastern Swim Club?
   b) What is the expected number of Eastern Swim Club members in the first heat?

9. The door prizes at a dance are four \$10 gift certificates, five \$20 gift certificates, and three \$50 gift certificates. The prize envelopes are mixed together in a bag, and five prizes are drawn at random.
   a) What is the probability that none of the prizes is a \$10 gift certificate?
   b) What is the expected number of \$20 gift certificates drawn?

10. A 12-member jury for a criminal case will be selected from a pool of 14 men and 11 women.

a) What is the probability that the jury will have 6 men and 6 women?

b) What is the probability that at least 3 jurors will be women?

c) What is the expected number of women?

11. Seven cards are dealt from a standard deck.

a) What is the probability that three of the seven cards are hearts?

b) What is the expected number of hearts?

12. A bag contains two red, five black, and four green marbles. Four marbles are selected at random, without replacement. Calculate

a) the probability that all four are black

b) the probability that exactly two are green

c) the probability that exactly two are green and none are red

d) the expected numbers of red, black, and green marbles

ACHIEVEMENT CHECK

Knowledge/ Understanding	Thinking/Inquiry/ Problem Solving	Communication	Application

13. A calculator manufacturer checks for defective products by testing 3 calculators out of every lot of 12. If a defective calculator is found, the lot is rejected.

a) Suppose 2 calculators in a lot are defective. Outline two ways of calculating the probability that the lot will be rejected. Calculate this probability.

b) The quality-control department wants to have at least a 30% chance of rejecting lots that contain only one defective calculator. Is testing 3 calculators in a lot of 12 sufficient? If not, how would you suggest they alter their quality-control techniques to achieve this standard? Support your answer with mathematical calculations.

C

14. Suppose you buy a lottery ticket for which you choose six different numbers between 1 and 40 inclusive. The order of the first five numbers is not important. The sixth number is a bonus number. To win first prize, all five regular numbers and the bonus number must match, respectively, the randomly generated winning numbers for the lottery. For the second prize, you must match the bonus number plus four of the regular numbers.

a) What is the probability of winning first prize?

b) What is the probability of winning second prize?

c) What is the probability of not winning a prize if your first three regular numbers match winning numbers?

15. **Inquiry/Problem Solving** Under what conditions would a binomial distribution be a good approximation for a hypergeometric distribution?

16. **Inquiry/Problem Solving** You start at a corner five blocks south and five blocks west of your friend. You walk north and east while your friend walks south and west at the same speed. What is the probability that the two of you will meet on your travels?

17. A research company has 50 employees, 20 of whom are over 40 years old. Of the 22 scientists on the staff, 12 are over 40. Compare the expected numbers of older and younger scientists in a randomly selected focus group of 10 employees.

Hypergeometric Distributions

A. What is a *Hyper*geometric Distribution?!

Unlike the other distributions we have encountered, **hypergeometric distributions** have trials whose probabilities ***do*** change from one trial to the next – making them **dependent** on one another!! Each dependent trial continues to have success or failure as the only options, but the probability of success changes as each trial is made. In a hypergeometric distribution,each trial reduces the number of items available for the next selection (*i.e.* items are not replaced/repeated).

The random variable for a hypergeometric distribution is number of successful trials in an experiment.

Examples: a) When choosing a starting lineup for a game,the coach must obviously choose a *different* player for each position.
b) When you deal cards from a standard deck, there can be *no repetitions*.
c) When choosing members to form a committee, there can also be *no repetition*.

Probability in a Hypergeometric Distribution

The probability of x successes in r **dependent** trials is given by the formula:

$$P(x) = \frac{({}_{a}C_{x})({}_{n-a}C_{r-x})}{{}_{n}C_{r}}$$

Where a is the number of successful outcomes among a total of n possible outcomes

The **hypergeometric probability distribution** can be created by finding the probabilities for all possible values of x .

Ex. 1: A committee of 5 is to be chosen from 6 men and 8 women.

(a) Determine the probability distribution for the number of women on this committee.

(b) What is the expected number of women on the committee?

Number of Women, *x*	**Probability, *P(x)***

Product of $x_i * P(x_i)$
E(X):

B. Expected Values

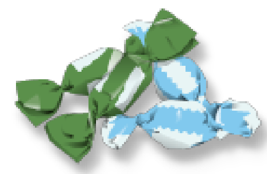
The **expected value,** *E(X),* for a hypergeometric distribution can be calculated "traditionally" by taking the sum of all $x_i * P(x_i)$, but it can also be simplified as a *ratio* of successes in the overall population as follows:

The expected value for a hypergeometric distribution for *r* dependent trials is given by the formula:

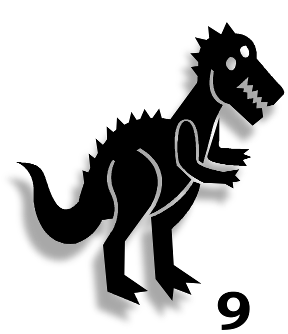
$$E(x) = \frac{r\,a}{n}$$

Where *a* is the number of successful outcomes among a total of n possible outcomes

Ex. 2: A box contains seven yellow, three green, five purple, and six red candies jumbled together. What is the expected number of red candies among five candies poured from the box?



Ex. 3: In the spring, 500 dinosaurs were caught in a wilderness area. The dinosaurs were released after being outfitted with laser vision. To estimate the dinosaur population in the area, rangers caught 40 dinosaurs in the summer. Of these, 15 had laser vision. Estimate the dinosaur population in the wilderness area.



Complete *Example #1* on p. 398-400 using Excel;   HW: p. 404-405 # 1-4, 6-8, 10-12

Practise

A

1. Which of these random variables have a hypergeometric distribution? Explain why.

a) the number of clubs dealt from a deck

b) the number of attempts before rolling a six with a die

c) the number of 3s produced by a random-number generator

d) the number of defective screws in a random sample of 20 taken from a production line that has a 2% defect rate

e) the number of male names on a page selected at random from a telephone book

f) the number of left-handed people in a group selected from the general population

g) the number of left-handed people selected from a group comprised equally of left-handed and right-handed people

2. Prepare a table and a graph of a hypergeometric distribution with

a) $n = 6, r = 3, a = 3$ **b)** $n = 8, r = 3, a = 5$

Apply, Solve, Communicate

B

3. There are five cats and seven dogs in a pet shop. Four pets are chosen at random for a visit to a children's hospital.

a) What is the probability that exactly two of the pets will be dogs?

b) What is the expected number of dogs chosen?

4. Communication Earlier this year, 520 seals were caught and tagged. On a recent survey, 30 out of 125 seals had been tagged.

a) Estimate the size of the seal population.

b) Explain why you cannot calculate the exact size of the seal population.

5. Of the 60 grade-12 students at a school, 45 are taking English. Suppose that 8 grade-12 students are selected at random for a survey.

a) Develop a simulation to determine the probability that 5 of the selected students are studying English.

b) Use the formulas developed in this section to verify your simulation results.

6. Inquiry/Problem Solving In a study of Canada geese, 200 of a known population of 1200 geese were caught and tagged. Later, another 50 geese were caught.

a) Develop a simulation to determine the expected number of tagged geese in the second sample.

b) Use the formulas developed in this section to verify your simulation results.

7. Application In a mathematics class of 20 students, 5 are bilingual. If the class is randomly divided into 4 project teams,

a) what is the probability that a team has fewer than 2 bilingual students?

b) what is the expected number of bilingual students on a team?

8. In a swim meet, there are 16 competitors, 5 of whom are from the Eastern Swim Club.

a) What is the probability that 2 of the 5 swimmers in the first heat are from the Eastern Swim Club?

b) What is the expected number of Eastern Swim Club members in the first heat?

9. The door prizes at a dance are four \$10 gift certificates, five \$20 gift certificates, and three \$50 gift certificates. The prize envelopes are mixed together in a bag, and five prizes are drawn at random.

a) What is the probability that none of the prizes is a \$10 gift certificate?

b) What is the expected number of \$20 gift certificates drawn?

10. A 12-member jury for a criminal case will be selected from a pool of 14 men and 11 women.

a) What is the probability that the jury will have 6 men and 6 women?

b) What is the probability that at least 3 jurors will be women?

c) What is the expected number of women?

11. Seven cards are dealt from a standard deck.

a) What is the probability that three of the seven cards are hearts?

b) What is the expected number of hearts?

12. A bag contains two red, five black, and four green marbles. Four marbles are selected at random, without replacement. Calculate

a) the probability that all four are black

b) the probability that exactly two are green

c) the probability that exactly two are green and none are red

d) the expected numbers of red, black, and green marbles

ACHIEVEMENT CHECK

Knowledge/ Understanding	Thinking/Inquiry/ Problem Solving	Communication	Application

13. A calculator manufacturer checks for defective products by testing 3 calculators out of every lot of 12. If a defective calculator is found, the lot is rejected.

a) Suppose 2 calculators in a lot are defective. Outline two ways of calculating the probability that the lot will be rejected. Calculate this probability.

b) The quality-control department wants to have at least a 30% chance of rejecting lots that contain only one defective calculator. Is testing 3 calculators in a lot of 12 sufficient? If not, how would you suggest they alter their quality-control techniques to achieve this standard? Support your answer with mathematical calculations.

C

14. Suppose you buy a lottery ticket for which you choose six different numbers between 1 and 40 inclusive. The order of the first five numbers is not important. The sixth number is a bonus number. To win first prize, all five regular numbers and the bonus number must match, respectively, the randomly generated winning numbers for the lottery. For the second prize, you must match the bonus number plus four of the regular numbers.

a) What is the probability of winning first prize?

b) What is the probability of winning second prize?

c) What is the probability of not winning a prize if your first three regular numbers match winning numbers?

15. Inquiry/Problem Solving Under what conditions would a binomial distribution be a good approximation for a hypergeometric distribution?

16. Inquiry/Problem Solving You start at a corner five blocks south and five blocks west of your friend. You walk north and east while your friend walks south and west at the same speed. What is the probability that the two of you will meet on your travels?

17. A research company has 50 employees, 20 of whom are over 40 years old. Of the 22 scientists on the staff, 12 are over 40. Compare the expected numbers of older and younger scientists in a randomly selected focus group of 10 employees.

Comparing Discrete Probability Distributions

Objectives for this lesson:

- compare and make connections between uniform, binomial, geometric, and hypergeometric distributions
- define discrete random variables and calculate their expected values
- use Excel to create tables and graphs for discrete probability distributions

Practice Problems:

1. A random draw lottery has sold $52 000 000 worth of tickets, with each ticket sold for $5.00. The prizes and their frequencies are shown in the following table.

Prize ($), *x*	**Number of Prizes**	***P(x)***
12 000 000.00	1	
206 955.00	5	
2 408.00	25	
147.00	100	
100.00	1 000	
10.00	5 000	

a) What is the random variable, X, for this distribution?
b) How many tickets were sold?
c) Complete the column *P(x)*.
d) What is the expected prize per ticket?
e) What is the expected profit per ticket for the lottery corporation?

2. In a family with three children:

a) Create a probability distribution in Excel for the number of boys. Define your random variable.
b) Graph the distribution.
c) Calculate the expected number of boys.

3. At the Toyota plant in Cambridge, an assembly line robot installs CD players in each car produced on the line. For quality control, the units are tested to make sure they are installed properly. The probability of a malfunction is 0.2.

a) Determine the probability that 5 CD players will be tested before a malfunction is found.
b) What is the expected number of tests before a malfunction is found?

4. A bag contains six red marbles, five yellow marbles and four black marbles. Marbles are selected from the bag, one at a time, without replacement until four marbles have been taken.

a) Create a probability distribution for choosing red marbles.
b) Graph your distribution.
c) Determine the expected number of red marbles.

5. Martin is virtually unbeatable in Mario Kart. For a challenge he decides to take on competitors and play with only one hand. He figures he still has a 75% chance of winning, no matter the opponent.

a) Assuming his probability of winning stays constant, create a table showing the probability distribution for the number of games Martin might win in five.
b) What is the probability of Martin winning more than two games?
c) What is the expected number of games Martin will win?

Probability Distributions – Group Study Activity

In groups of 2, 3 or 4 solve the following permutation problems. Write full solutions and check your answers against the key provided.

1. At the Toyota plant in Cambridge, an assembly line robot installs CD players in each car produced on the line. For quality control, the units are tested to make sure they are installed properly. The robot has a probability of malfunctioning of 0.2.
   a) Determine the probability distribution of the number of tests performed before a malfunction is found. (Only complete the distribution for $0.2 \geq P \geq 0.09$)
   b) What is the expected number of tests before a malfunction is found?

2. A school lottery has 5000 tickets for sale. There is one grand prize of $2000 and five second prizes of $100.
   a) Determine the expected value of this lottery and interpret what it represents.
   b) If you buy a ticket for $1, what is your expected value?
   c) How can the school use the expected value to help determine the ticket price?

3. A bag contains six red marbles, five yellow marbles and four black marbles. Marbles are selected from the bag, one at a time, without replacement until four marbles have been taken.
   a) Determine the probability distribution for choosing red marbles.
   b) Determine the expected number of red marbles.

4. Conner has five shots in a shoot-out competition. Historically, Conner has a probability of 35% (a very high probability) of scoring on a shoot-out.
   a) Assuming this probability stays constant, create a table showing the probability distribution and the expected number of goals Conner might score.
   b) What is the probability of Conner scoring more than two goals?
   c) What is the expected number of goals Conner will score?

Answer key

1b) 4	2a) $0.50	2b) – $0.50	3b) 1.6	4b) 23.4%
4c) 1.75				

SELF QUIZ

1. The eighteenth hole at a public golf course is a par 4. This means that a good golfer should be able to play the hole in 4 strokes. The table below summarizes the scores of golfers who recently played the hole.

Number of Strokes, x	Frequency	Probability, $P(x)$	$x * P(x)$
1	0		
2	3		
3	24		
4	45		
5	51		
6	38		
7	24		
8	9		
		$E(x) =$	

a) Complete the Probability Distribution chart for the number of strokes above.

b) What is the expected value for the number of strokes a golfer will take on this hole? _________

c) Is it reasonable that the expected value for the hole should differ from the par value? Explain.

2. The Nestle Candy Company manufactures Smarties, a candy coated chocolate. On average, 40% of all Smarties are red. For the Hallowe'en sized boxes, the production line mixes the candies randomly and packages ten per box.

   a) What is the probability that *less than* four candies in a given box are red?

   b) What is the probability that *at least* four candies in a given box are red?

   c) What is the expected number of red candies per box?

3. For a symmetrical eight-sided die,

   a) what is the probability that it will take exactly two tries to roll a 7?

   b) what is the expected waiting time until a 7 is rolled?